

Crises and Physical Phases of a Bipartite Market Model

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We analyze the linear response of a market network to shocks based on the bipartite market model we introduced in [10], which we claimed to be able to identify the time-line of the 2009-2011 Eurozone crisis correctly. We show that this model has three distinct phases that can broadly be categorized as “stable” and “unstable”. Based on the interpretation of our behavioral parameters, the stable phase describes periods where investors and traders have confidence in the market (e.g. predict that the market rebounds from a loss). We show that the unstable phase happens when there is a lack of confidence and seems to describe “boom-bust” periods in which changes in prices are exponential. We analytically derive these phases and where the phase transition happens using a mean field approximation of the model. We show that the condition for stability is $\alpha\beta < 1$ with α being the inverse of the “price elasticity” and β the “income elasticity of demand”, which measures how rash the investors make decisions. We also show that in the mean-field limit this model reduces to a Langevin model [6] for price returns. Thus we provide analytical support for the power of the model in classifying the stability of markets where it’s applicable.

I. INTRODUCTION

Economical and financial systems are attractive to physicists because of their highly dynamic and complex structure which is generally much more challenging as a complex system than systems exhibiting a high degree of symmetry studied in traditional physics. The goal of this paper is to derive analytical results regarding the physics and phases of a model we developed for quantifying the linear response in a bipartite, dynamical network. While our discussion will exclusively concern a largely simplified financial network and its mean-field behavior, a similar methodology based on an “effective theory” approach could be employed in analyzing the response of many other dynamical bipartite networks. Especially networks pertaining to resources and agents using resources, such as power grids and some special food webs, might benefit from a similar type of modeling.

The global financial crisis that followed which started around 2007 made it clear to us how limited our understanding of the dynamics of financial markets are. In recent years, economists seem to have acknowledgement the fact that we cannot predict the real world dynamics with idealized assumptions and we need to account for complex relations of different players in an economical system. Scholars have started consider the network of such interactions and how they may affect business cycles, cause cascades, or contagion. [1, 2]. Our goal here is to make a first attempt at quantitatively modeling the continuous dynamics of a market system in the simplest form and to lowest order. In an earlier paper we introduced a bipartite network model for investment markets in which investors traded assets in a fashion similar to common stock markets [10]. The assumption of the mar-

ket being a bipartite network is, of course, a major simplification. In reality, there undoubtedly exist many strong financial interactions among investors and even among assets. In our paper [10] the major investors were banks or other major financial institutions. It should be noted that interbank lending network is a very complex and important, multiplex network and much research has been dedicated to it (see [4, 12, 13, 19] and references therein).

Our simplified bipartite model, which has “assets” as one layer and “investors” on the other side, ignoring all intra-layer connections, was constructed in order to assess the first order response dynamics of a market to any change. In this model neither money nor number of shares is conserved, as investors are assumed to trade with entities both inside and outside the network – though such conservation laws can be imposed if needed. The weighted connections indicate the amount of investments. The details of the model and the phenomenological derivation for it are explained in the [10]. The goal of this paper is to derive analytical results about the stability conditions for such market dynamics.

There are many different models for financial networks [3, 14–17] each assessing a different aspects of connectivity which use familiar assumptions about dynamics in of prices based on existing economic models. Our model, on the other hand, is at its core what we call an “effective theory” in physics. It attempts to model the dynamics of a market by making minimal assumptions about details and assuming system-wide parameters. It only uses the microscopic behavior of agents as input. The demonstration of the genericity of our model is the subject of another forthcoming paper [8].

Perhaps the closest model to ours in the context bipartite model of investment networks was introduced by Caccioli et al. [7]. As will become clear in the deriva-

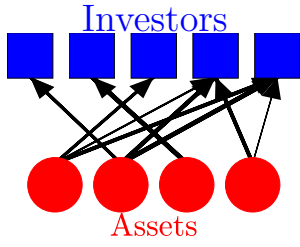


FIG. 1. A sketch of the network of investors vs assets. It is a directed, weighted bipartite graph (thicknesses represent weights of investment). The weighted adjacency matrix has entries $A_{i\mu}$, the number of shares of asset μ held by investor i .

tions below, our model in fact also reduces to a Langevin equation's model for price-dynamics in a stock market introduced by Bouchaud [6]. There the authors argue that the price dynamics near the onset of a crisis is similar to a damped harmonic oscillator plus non-linear terms that become relevant after the phase transition to an unstable phase. Our model has such non-linear naturally from the start and no extra assumption is needed. In another forthcoming paper we will discuss the application of our model in stock markets in detail.

Our model relies on 2 behavioral parameters, one for supply/demand and one for investor decisions. Over time-scales where the behavioral parameters are not changing much, we are arguing that the large scale dynamics of the market can be modeled and simulated. It is in this regime that our model may provide crucial insight into stability of a market and serve as a tool for regulators.

II. MODEL AND NOTATION

We will refer the model as the Group-Impulse Portfolio-Sharing Investment (GIPSI) model henceforth. In essence the GIPSI model summarizes how investors and their brokers may behave on average in a market, if they are aware of the market news.

We will approximate the investment market as a bipartite network as shown in Fig. 1. On one side we have the “Assets,” which are what is traded, and on the other we have the “Investors” that own the assets. The “assets” are labeled using Greek indices μ, ν, \dots . To each asset μ we assign a “price,” $p_\mu(t)$ at time t . The “investors” are labeled using Roman indices i, j, \dots . Each investor has an “equity” $E_i(t)$, a time t , i.e. their net worth. Each investor has a portfolio, meaning differing amounts of holdings in each of the asset types. The amount of asset μ that investor i holds is denoted by $A_{i\mu}(t)$, which is essentially an entry of the weighted adjacency matrix A of the bipartite network.

We assume that there is a group-impulse in the

decision-making, the so-called “herding effect” [5, 11] where all agents will have the same level of panic or calmness in a situation. They will adjust their portfolio $A_{i\mu}$ by looking at the gains or losses δE_i they incurred recently. So one of the equations is a generalization of a simple portfolio adjustment protocol

$$\delta \log A_{i\mu} \sim \beta \delta \log E_i$$

We argue that, if the assumption Their level of panic is reflected in the so-called “Income elasticity of demand”, which we will denote as β . This assumption has been criticized by some scholars as being unrealistic based on other empirical evidence about “fire sales” [18], claiming that the leverage (i.e. ratio of investments to equity) . Our justification for this assumption is based on a mean field assumption. We believe that most of these institutions will be leveraged to close to the maximum amount allowed by regulations, thus having similar leverage. Aside from that, if the assumption of herding holds for response to a rapid change, using the same factor β makes sense for at least the mean field behavior of the system.

The supply and demand equation for the prices p_μ has a similar structure

$$\delta \log p_\mu \sim \alpha \delta \log A_{i\mu}$$

where $1/\alpha$ would be what is usually called “price elasticity” in economics. There is also a third equation which is related to how trading, i.e. $\delta(A \cdot p)_i$ changes the equity of an investor, which turns out to be [10] $\delta E_i = (A \cdot \delta p)_i$. A crucial point for having a more realistic model for such systems is the fact that there is a response time associated with each of these equations. describing how each of the variables $E_i(t)$, $A_{i\mu}(t)$, and $p_\mu(t)$ evolve over time. A key feature of our model is that the weights of links $A_{i\mu}$ are time-dependent, and this introduces dynamics into our network.

For brevity, we define $\partial_t \equiv \frac{d}{dt}$. The equations of the GIPSI model may be written as

$$(\tau_B \partial_t^2 + \partial_t) A_{i\mu}(t) = \beta \frac{\partial_t E_i(t)}{E_i(t)} A_{i\mu}(t) \quad (1)$$

$$(\tau_A \partial_t^2 + \partial_t) p_\mu(t) = \alpha \frac{\partial_t A_{i\mu}(t)}{A_{i\mu}(t)} p_\mu(t) \quad (2)$$

$$\partial_t E_i(t) = \sum_\mu A_{i\mu}(t) \partial_t p_\mu(t) + f_i(t). \quad (3)$$

where $f_i = dS_i/dt$ has the meaning of external force. where τ_B is the time-scale in which investors respond to a change in their net worth, and τ_A is the time-scale of market's response.

The importance of this model as a lowest order approximation of linear response in a market system is the subject of a separate paper in preparation [8, 9]. But just to motivate the use of this model, we only note that the lowest order effective Lagrangian model (in the spirit of

TABLE I. Notation

symbol	denotes
$A_{i\mu}(t)$	Holdings of investor i in asset μ at time t
$p_\mu(t)$	Normalized price of asset μ at time t ($p_\mu(0) = 1$)
$E_i(t)$	Equity of bank i at time t .
α	Inverse price elasticity
β	Income elasticity of demand (rashness)

Landau-Ginzburg models) for the response of a market defined by the dynamical variables $A_{i\mu}$, p_μ and E_i plus dissipation yields equations with the structure of (1)-(3).

III. CONFIDENCE IN THE MARKET IN TERMS OF BASED ON MARKET RESPONSE

The simple strategies devised above are expected to describe the linear response of the market in the mean-field approximation. The sign of β may be considered an indicator for confidence in the market: positive β means in response to a loss (i.e. $\delta E_i < 0$) the investor sells assets (i.e. $\delta A_{i\mu} < 0$). This can be interpreted as the investor fearing more losses and therefore reducing their holdings. By the same token negative β may signal confidence in the market as it indicates a willingness to buy more shares when the investor has lost money.

Note that the GIPSI model Eq. (1)–(3) are response equation, i.e. they yield no dynamics of there is no change in the variables E, A, p . To assess the behavior of this system we will introduce an initial shock to the system. We do this by assuming a delta function change in equity $f_i(t) = c\delta(t)$ for some investor i , meaning that at $t = 0$ agent i either gained or lost money from outside the market and it's decision to trade in the market triggers the response of the market ¹.

For the empirical analysis we will use the same Eurozone crisis data we used in the original paper [10] which consists of sovereign bonds of five European countries, namely Greece Italy, Ireland, Portugal and Spain (GIIPS), as assets and the network of major banks and financial institutions who purchased and traded these bonds. Many smaller players and the European Central Bank (ECB) that also participated in the trades are not in this data and their existence justifies the non-conservation of the total amount of holdings in the network. They are

also partly responsible for dissipation in the dynamics. The data is given in the appendix of [10].

Fig. 2 shows 4 examples of choices for α and β and the response of the prices and equities of banks to a shock to a random bank in the system ². As we will show below, all these choices fall in the “stable” regime, though the losses incurred from a negative shock (i.e. $f_i < 0$) are much larger in magnitude when α and β have the same sign than when they have opposite signs. The upper left plot in Fig. 2 is for α and $\beta > 0$. This is what one normally expects from this system: $\beta > 0$ means if a bank incurs a loss, they try to make up for it by making money from selling assets; $\alpha > 0$ means if there is selling pressure (more supply than demand) the prices will go down. There are, however, cases where the opposite happens. Negative values for α and β are possible in the real world and are known as “contrarian” behavior. For example, a contrarian investor is someone who invest more in asset μ when they actually lose money on asset μ . This happens a lot when there is confidence in the market and investors believe the price will rebound. The market may also sometimes behave in a contrarian fashion, when there is an anticipation of good news that overcomes the selling pressure, or when other investors outside our Eurozone GIIPS network (such as smaller investors or the ECB) are actually exerting a buying pressure.

As is seen in Fig. 2 this results in price slightly oscillating around its original value. This trust mechanism plays an important role in the stability of the market. It requires that one side (either investor or the prices) behaves in a contrarian fashion and the other side behaves normally.

IV. PHASES AND CONFIDENCE

As we see the behavior of prices p_μ and equities E_i in the case where the product $\gamma \equiv \alpha\beta$ is positive and negative differ qualitatively. When $\gamma > 0$ negative shocks ($\partial_t E_i(0) < 0$) cause a noticeable drop in prices and some equities³, whereas when $\gamma < 0$ there is only oscillation and the final values of p_μ and E_i are very close to their original values. There is, however, another qualitatively different regime which is not shown in Fig. 2. This happens when $\gamma > 1$ and is shown in Fig. 3. In fact, this qualitative difference becomes much clearer when we show the effect of a positive shock $\partial_t E_i(0) > 0$ to the system in Fig. 4. There we see that when $\gamma < 1$ after a positive shock eventually stops growing and reaches a

¹ It turns out that the qualitative behavior of the final state after a shock to the system is only weakly sensitive to the value of the response times τ_A, τ_B . The response times only need to be nonzero. As we will show analytically below, the phase of the system is determined by the two behavioral couplings α and β . The plots shown below are for $\tau_A = \tau_B = 1$.

² It turns out that, similar to a system in the thermodynamic limit, the details of which bank is shocked through f_i doesn't matter in the final state of the system [10].

³ All of E, A, p are positive quantities and we do not allow them to go negative.

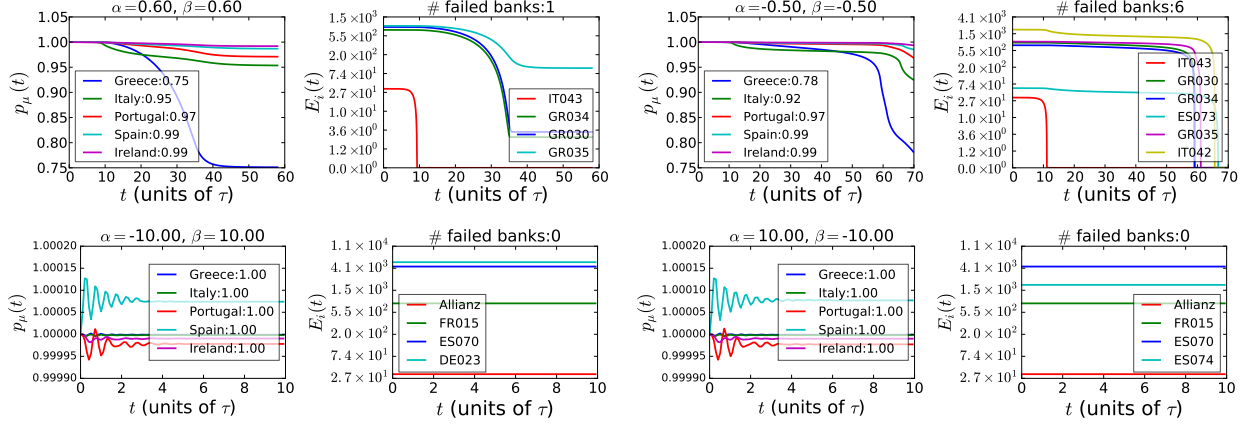


FIG. 2. Contrarian regimes: top, both $\alpha, \beta < 0$. Here many banks fail, even for relatively small α, β . The losses are devastating. Our model suggests that such a regime should be avoided. The bottom two plots show the two points $\alpha = \pm\beta, \beta = \pm 10$. The two results are almost identical. They also show that no appreciable amount of profit or loss is generated in these regimes, thus making them rather unfavorable for investors most of the time, but because of their safeness could be a contingency plan (buyout of bad assets by central banks is one such contrarian behavior).

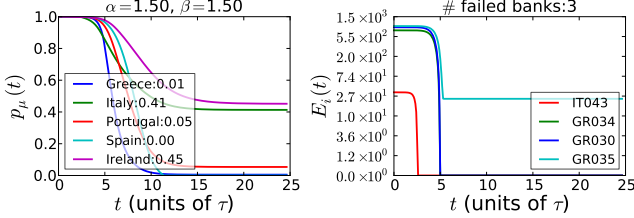


FIG. 3. When $\gamma = \alpha\beta > 1$, although the shape of the price and equity plot after a negative shock look similar to when $1 > \gamma > 0$, the behavior of the system is qualitatively different. Many asset prices plummet down to or close to zero. The system survives only because investors (banks) whose equity goes to zero stop existing and won't propagate the shock further.

new equilibrium, whereas when $\gamma > 1$ all three variables E, A, p grow exponentially, indefinitely. In other word an economic “bubble” forms. So $\gamma > 1$ if it lasts for a long time either forms a bubble or results in a crash.

This leads to the following conclusions:

1. When $\gamma = \alpha\beta < 0$ no investor goes bankrupt, but also the amount of money lost or generated during the trading is negligible. This makes these regimes (where either the investors or the market is contrarian, but not both) good for preventing failures, but they are very undesirable for profit making.
2. When $1 > \gamma > 0$ the system does not show oscillation but eventually settles into new equilibrium states not far from the original situation. Negative shocks may cause bankruptcy of some investors, depending on how their assets $(A \cdot p)_i$ compares to their equity E_i initially.

3. When $\gamma > 1$, negative shocks cause exponential drops in assets and equities. If $\gamma < 1$ persists there will be a crisis. Positive shocks may form bubbles and periods of exponential growth.

A. The full phase diagram

Fig. 5 shows an example of the average final prices and relaxation time for the system for various values of α and β . It seems the system has two prominent phases: One in which a new equilibrium is reached without a significant depreciation in all of the GIIPS holdings (upper left and lower right quadrants), and one where all GIIPS holdings become worthless (above dashed line in the upper right quadrant and all of lower left quadrant). In both the first and the third quadrants in the transition region the relaxation time becomes very large, which means that the forces driving the dynamics become very weak. Both the smoothness and the relaxation time growth seem to be signalling the existence of a second order phase transition. The phase transition seems to be described well by:

$$\gamma = \alpha\beta = 1.$$

Now we do a systematic numerical analysis of different phases of this phenomenological model. We identify to phases and what appears to be a second order phase transition between them. We then modify the equations (1)–(3) and analytically derive the condition for the phase transition. The GIIPS data is very heterogeneous, in the sense that for each of these European countries there are generally less than a handful of investors who hold more than half of the total sovereign debt of that country. If

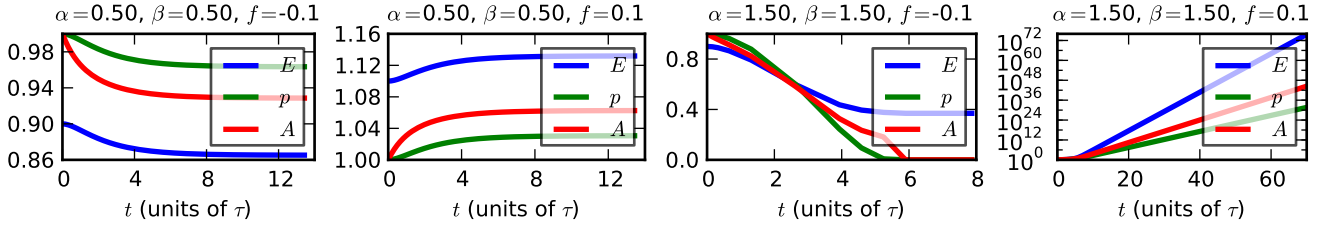


FIG. 4. Numerical solutions to the differential equations in a 1 bank vs 1 asset system. The upper plots show a “stable” regime, where after the shock none of the variables decays to zero or blows up, but rather asymptotes to a new set of values. The lower plots are in the “unstable” regime where positive or negative shocks either result in collapse or blowing up or collapsing of some variables.

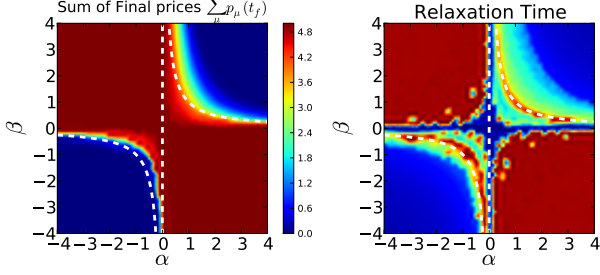


FIG. 5. **Left:** Phase diagram of the GIIPS sovereign debt data, using the sum of the final price ratios as the order parameter. We can see a clear change in the phase diagram from the red phase, where the average final price is high to the blue phase, where it drops to zero. The drop to the blue phase is more sudden in the $\alpha < 0, \beta < 0$ quadrant than the first quadrant. **Right:** The time it takes for the system to reach the new equilibrium phase. This relaxation time significantly increases around the transition region, which supports the idea that a phase transition (apparently second order) could be happening in the first and third quadrants. The dashed white line shows the curve $\gamma = \alpha\beta = 1$. It fits the red curves of long relaxation time very well. This may suggest that $\gamma = 1$ is a critical value which separates two phases of the system.

we make a mean field assumption and break the network apart, assuming there is one major investor for each asset μ , we can treat that as a 1 investor 1 asset system and try to derive the phases in that case. As we show below, the 1 by 1 system, despite not having the richness of the networked system in terms of the details of the final state, nevertheless exhibits the same phases. In addition we are able to analytically derive the phase transitions.

V. ANALYTICAL DERIVATION OF THE MEAN-FIELD PHASE SPACE

In the mean-field approach described above, we are dealing with a 1 investor by 1 asset system with a single E, A and p . We can combine Eqs. (1)–(3) in a 1 by 1 system by taking another ∂_t derivative from (2). This way we can eliminate most occurrences of E and A and

find an equation for p (see Appendix A) which has some non-linear terms in it.

Defining the “return” $u \equiv \partial_t p$ as the fundamental variable, the nonlinearities are roughly of type $u^2 + a\partial_t u^2$. In short, the equations are

$$\begin{aligned} [\tau \partial_t^2 + \partial_t + \omega^2] u &= \frac{O(u^2, \partial_t u^2)}{p} \\ \frac{1}{\tau} &= \frac{1}{\tau_B} + \frac{1}{\tau_A}, \quad \omega^2 = \frac{1 - \gamma \frac{Ap}{E}}{\tau_B + \tau_A}. \end{aligned} \quad (4)$$

As we show in Appendix A none of the nonlinearities on the r.h.s. can be large in the stable phase where $\gamma < 1$. We also show there that Ap/E becomes its $t = 0$ value plus other nonlinear terms that cannot be large when $\gamma < 1$ and $\frac{Ap}{E} \approx 1 + f_0 + O(u^2)$. For a small shock $f_0 = -\varepsilon$ we may safely use $Ap/E = 1$. Thus for a time-scale where $\gamma < 1$ and is not changing much we are essentially dealing with a damped harmonic oscillator. Notice that equation (4) is almost identical to what Bouchaud proposes in [6] to explain the 1987 crash.

Although ω^2 depends on A, p and E , we can use an approximate exponential ansatz $u \sim u_0 \exp[\lambda t]$. The solutions to λ are:

$$\lambda_{\pm} = \frac{-1 \pm \sqrt{1 - 4\tau\omega^2}}{2\tau}$$

Thus we get three regimes:

1. When $\omega^2 > \frac{1}{4\tau}$ there will be oscillatory solutions. This happens when $\gamma < \frac{-(\tau_A - \tau_B)^2}{\tau_A \tau_B}$. For $\tau_A = \tau_B$ this is just the $\gamma < 0$ condition we observed for oscillations in our simulations.
2. When $\frac{1}{4\tau} > \omega^2 > 0$, i.e. $\frac{-(\tau_A - \tau_B)^2}{\tau_A \tau_B} < \gamma < 1$ we have decaying solutions but both $\lambda_{\pm} < 0$. Therefore the changes won’t be large and eventually the system settles in new equilibrium.
3. When $\omega^2 < 0$, i.e. $\gamma > 1$ we will have two real solutions for with opposite signs. The presence of the positive root $\lambda_+ > 0$ signals an instability because this solution diverges.

Thus we have proven the existence of the three phases we had observed earlier and derived the transition conditions analytically. It is easy to prove that in the unstable phase both exponents λ_{\pm} appear in the solution: The return $\partial_t p$ is

$$\partial_t p(t) = u(t) = u_+ e^{\lambda_+ t} + u_- e^{\lambda_- t}$$

Since at $t = 0$ the initial conditions dictated $\partial_t p(0) = 0$ we have $u_+ = -u_-$ and therefore both solutions appear with equal strength. It follows that whenever one of the solutions (u_- in our case) is positive the solution diverges. When $f > 0$ a bubble forms and grows exponentially and when $f < 0$, because our variables are non-negative, the price just crashes to zero. This proves that the sufficient condition for stability is $\gamma < 1$. Also note that the non-linear terms are all proportional to $\partial_t p$ and therefore at $t = 0$

$$O(u^2(0), \partial_t u^2(0)) = 0$$

and so the solution is exact at $t = 0$. The details of the derivations as well as the nonlinear terms are given in Appendix A.

In Appendix B we provide another, direct proof for the phase transition being at $\gamma = 1$ using exponential ansatz for all three variables $X = E, p, A$

$$X \sim X_0 + X_1 \exp[w_{X1}t] + X_2 \exp[w_{X2}t].$$

We then show that near the phase transition the exponents in E, A, p show that the near the phase exponents for all three need to be the same. Moreover, as can be seen in the right figure in Fig. 5 near the phase transition the time-scale of the system's evolution is very long compared to other time-scales in the system and thus much longer than τ in (4), meaning

$$|w_X| \ll \tau^{-1}$$

This leads to a massive simplification of the equations and yields

$$\gamma = \frac{1 + f_0}{1 - \beta f_0}$$

as the phase transition point, which for infinitesimal shocks $f_0 \rightarrow 0$ recovers $\gamma = 1$.

VI. CONCLUSION

We have shown that the GIPSI model [10] for response dynamics of a bipartite investment market exhibits three phases: two stable phases, one of which is oscillatory the other logistic-like; one unstable phase with indefinite exponential decay or growth depending on the initial perturbation. The unstable phase can exist in real world,

though it must be short-lived. It can describe “boom-bust” periods: crisis periods (as was shown in [10]) or periods of rapid growth, so-called “bubbles”. We also derive these phases analytically from a mean-field version of the model. The mean-field equations become almost identical to the Langevin equations proposed by Bouchaud [6] which could well describe behavior of markets near crashes. The non-linear terms required for the transition in Bouchaud’s model exist naturally in the GIPSI model and nothing new needs to be added by hand.

VII. ACKNOWLEDGMENTS

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Appendix A: Analytical results from the 1 Investor vs 1 Asset system

Here we present the analytical solution to the 1 by 1 model and derive the curve where the phase transition is happening in figure 5. At any time t the equations for a 1 by 1 system become

$$\begin{aligned}\frac{(\partial_t + \tau_B \partial_t^2)A}{A} &= \beta \frac{\partial_t E}{E} = \beta \frac{A \partial_t p}{E} \\ \frac{(\partial_t + \tau_A \partial_t^2)p}{p} &= \alpha \frac{\partial_t A}{A}\end{aligned}\quad (A1)$$

Below we will try to find the condition for a phase transition in the solutions to these equations.

We can try to eliminate A and E . We first need to find the expression for $\partial_t^2 A/A$ first. Taking another derivative from the second equation yields

$$\frac{(\partial_t^2 + \tau_A \partial_t^3)p}{p} - \frac{(\partial_t + \tau_A \partial_t^2)p \partial_t p}{p^2} = \alpha \frac{\partial_t^2 A}{A} - \alpha \left(\frac{\partial_t A}{A} \right)^2 \quad (A2)$$

combining this with the first equation results in:

$$\begin{aligned}\frac{(\partial_t + \tau_A \partial_t^2)p}{p} + \tau_B \frac{(\partial_t^2 + \tau_A \partial_t^3)p}{p} &= \gamma \frac{A \partial_t p}{E} + O((\partial_t p)^2) \\ \left[\tau_A \tau_B \partial_t^2 + (\tau_A + \tau_B) \partial_t + \left(1 - \gamma \frac{Ap}{E} \right) \right] \partial_t p &= O((\partial_t p)^2)\end{aligned}\quad (A3)$$

where the nonlinear term is again quadratic in p (thus a generalized form of the Fisher equation) and looks like

$$\begin{aligned}O((\partial_t p)^2) &= \tau_B \frac{(1 + \tau_A \partial_t) \partial_t p \partial_t p}{p} \\ &\quad - \alpha \tau_B \frac{((1 + \tau_A \partial_t) \partial_t p)^2}{p}\end{aligned}\quad (A4)$$

Below we will also show that in the stable regime the non-linearity in the frequency, namely the $\gamma Ap/E$ term, is of the order $O(\partial_t A \partial_t p)$ and thus remains small if we show that at small times the behavior of $\partial_t p$ in the stable regime is oscillating around zero.

This time the dynamics is richer and we have a damped oscillator with a driving force coupled to p and nonlinearities of type $\sim (\partial_t p)^2$. Taking the return $u \equiv \partial_t p$ as the fundamental variable, the nonlinearities are roughly of type $u^2 + a \partial_t u^2$. In short, the equations are

$$\begin{aligned}[\tau \partial_t^2 + \partial_t + \omega^2] u &= O(u^2, \partial_t u^2) \\ \frac{1}{\tau} &= \frac{1}{\tau_B} + \frac{1}{\tau_A}, \quad \omega^2 = \frac{1 - \gamma \frac{Ap}{E}}{\tau_B + \tau_A}, \\ p(t) &= \int^t u(t') dt'\end{aligned}\quad (A5)$$

Although ω^2 depends on A, p and E , we can use an approximate time dependent exponential ansatz $u \sim u_0 \exp[\lambda t]$. The solutions to λ are:

$$\lambda_{\pm} = \frac{-1 \pm \sqrt{1 - 4\tau\omega^2}}{2\tau}$$

When $\omega^2 > 0$ and $1 - 4\tau\omega^2 < 0$ there will be oscillatory solutions. For example when $\gamma \frac{Ap}{E} < -1$, which only happens for negative γ we have such oscillatory solutions. This is consistent with the simulations which showed the oscillatory behavior was in the $\alpha\beta < 0$ quadrants. For the stability, however we care about the real solutions.

When $\omega^2 < 0$, which happens when $\gamma \frac{Ap}{E} > 1$, we will have two real solutions with opposite signs. The presence of the positive root signals an instability because the solution diverges. For a delta function shock of magnitude f at $t = 0$ we found that:

$$E_0 \rightarrow E_0(1 + f)$$

Having initially scaled to $E_0 = A_0 = p_0 = 1$, the condition for existence of the positive root becomes:

$$t = 0: \quad \gamma > \frac{E}{Ap} = (1 + f)$$

This dependence on the shock magnitude is normal, as a strong enough kick can kick a particle out of a local minimum. The shock can be arbitrarily small and therefore the absolute condition for stability is as we anticipated

$$\text{unstable at: } \gamma > 1 \quad (A6)$$

Now the question is, which solution does the system pick when it is shocked. The return $\partial_t p$ is

$$\partial_t p(t) = u(t) = u_+ e^{\lambda_+ t} + u_- e^{\lambda_- t}$$

Since at $t = 0$ the initial conditions dictated $\partial_t p(0) = 0$ we have

$$u_+ = -u_-$$

And therefore both solutions appear with equal strength. It follows that whenever one of the solutions (u_- in our case) is positive the solution diverges. When $f > 0$ a bubble forms and grows exponentially and when $f < 0$, because our variables are non-negative, the price just crashes to zero. This proves that the sufficient condition for stability is $\gamma < 1$. Also note that the nonlinear terms are all proportional to $\partial_t p$ and therefore at $t = 0$

$$O(u^2(0), \partial_t u^2(0)) = 0$$

and so the solution is exact at $t = 0$.

1. Validity of perturbation theory near the phase transition

For the above solution to be valid we must confirm that the corrections are small. We must find a small parameter that exists in the neglected terms which allows perturbative solutions to be viable. We had two sets of nonlinearities: (1) $O((\partial_t p)^2)$; (2) $\gamma Ap/E$.

a. the non-linearity $O((\partial_t p)^2)$

First let us examine the nonlinear terms in $O((\partial_t p)^2)$. Note that the instability happens when the larger root λ_- becomes positive. Thus near the transition we have

$$\begin{aligned} 4\tau\omega^2 &\ll 1 \\ \lambda_+ &\approx -\frac{1}{\tau} + \omega^2 \\ \lambda_- &\approx -\omega^2 \end{aligned} \quad (\text{A7})$$

And so being close to the phase transition means $\lambda_- \ll 1/\tau$. The consequence of this is that for $O((\partial_t p)^2)$ we get (using the $u_+ = -u_-$ found above)

$$\begin{aligned} \tau_A \partial_t u &= \tau_A u_+ (\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}) \\ &\approx \tau_A u_+ (\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}) \\ O(u^2) &= \tau_B \frac{u(1 + \tau_A \partial_t u)}{p} \\ &\quad - \alpha \tau_B \frac{((1 + \tau_A \partial_t u)u)^2}{p} \\ &\approx \tau_B \frac{u(1 + \tau_A u_+ (\lambda_+ - \lambda_-))u}{p} \\ &\quad - \alpha \tau_B \frac{((1 + \tau_A \partial_t u)u)^2}{p} \end{aligned} \quad (\text{A8})$$

b. The non-linearity $\gamma Ap/E$

We wish to examine if the assumption that in the stable regime $\partial_t A, \partial_t p, \partial_E$ remain small is a consistent assumption, thus making perturbative expansion valid. Any term above non-linear in $\partial_t A, \partial_t p, \partial_E$ is thus higher order in this approximation. We wish to find the part of $\gamma Ap/E \partial_t p$ that is linear in the first time derivative. In the stable regime changes are slow and thus a short time after the shock we can expand the variables in Taylor series near $t = 0$. Again, we will rescale the variables at $t = 0$ to $E_0 = p_0 = A_0 = 1$. Using the (3) $\partial_t E = A \partial_t p$

we get

$$\begin{aligned} \frac{A(t)p(t)}{E(t)} \partial_t p &= \frac{A_0 p_0 + t(\partial_t A_0 p_0 + A_0 \partial_t p_0)}{E_0 + t A_0 \partial_t p_0} \partial_t p \\ &\approx \frac{1}{E_0} (A_0 p_0 + t(\partial_t A_0 p_0)) \partial_t p \\ &= \partial_t p + O(\partial_t A_0 \partial_t p) \approx \partial_t p \end{aligned} \quad (\text{A9})$$

Thus the assumption of smallness of the derivatives is consistent and we may use perturbation theory and safely discard the non-linear terms in finding the stability conditions. This way the stability condition is just having a positive ω^2 in (A6). One can also check the stability by explicitly using the exponential ansatz found above as is given in what follows.

Appendix B: Proof for $\gamma = 1$ using properties of the phase transition

Since we have coupled second order equations, the solutions may be estimated using an exponential ansatz as follows. Equations (1) and (2) are second order and therefore will naturally have two solutions for A and p . Also, since $\partial_t E = A \partial_t p$, E will also have two modes. Therefore the exponential ansatz must have at least two exponents. Thus for each of the three variables $X = E, p, A$ we have:

$$X \sim X_0 + X_1 \exp[w_{X1} t] + X_2 \exp[w_{X2} t]$$

In principle the exponents can be time-dependent, but we will first try and see if there are asymptotically exponential solutions. Thus we assume that they vary slowly with time. By choosing the units of E, p, A to be such that at $t = 0 + \epsilon$, $p = A = 1$ and the shocked equity is $E = 1 + f$, the boundary conditions that we had become:

$$A_0 = 1 - A_1 - A_2, \quad p_0 = 1 - p_1 - p_2, \quad E_0 = 1 + f - E_1 - E_2$$

and:

$$\begin{aligned} \partial_t E(0) &= 0 = w_{E1} E_1 + w_{E2} E_2 = 0 \\ &= A \partial_t p = w_{p1} p_1 + w_{p2} p_2 \\ \partial_t A(0) &= \frac{\beta}{\tau_B} \ln(1 + f) = w_{A1} A_1 + w_{A2} A_2 \end{aligned} \quad (\text{B1})$$

At the phase transition we expect the greater exponents, which we take to be w_{X2} , to become small relative to other time-scales in the problem, i.e. τ_A, τ_B , and change sign from negative (which would result in exponential decay) to positive (which results in divergence of E, p, A). This means that close to the phase transition:

$$|w_{X2}| \ll |w_{X1}|, \quad w_{X2} \ll \frac{1}{\tau_A} + \frac{1}{\tau_B}$$

From the initial conditions, this results in:

$$\begin{aligned}
|E_1| &= \left| \frac{w_{E2}}{w_{E1}} E_2 \right| \ll |E_2|, \\
\Rightarrow E_0 &= 1 + f - \left(1 - \frac{w_{E2}}{w_{E1}} \right) E_2 \approx 1 + f - E_2 \\
|p_1| &\ll |p_2|, \quad \Rightarrow p_0 \approx 1 - p_2
\end{aligned} \tag{B2}$$

For A we have a little more details.

$$A_1 = \frac{\beta}{w_{A1}\tau_B} \ln(1+f) - \frac{w_{A2}}{w_{A1}} A_2 \approx \frac{\beta}{w_{A1}\tau_B} \ln(1+f)$$

Which for small shocks $f \ll 1$ reduces to:

$$A_1 \approx \frac{\beta}{w_{A1}\tau_B} f$$

Now back to the equations (1)-(3). First let us reexamine the third equation (3). The effect of a delta function shock $f(t) = f_0 \delta(t)$ is the above $\partial_t A$ and $E(+\epsilon) = (1 + f_0)$. Since $|w_{X2}| \ll |w_{X1}|$ and $w_{X1} < 0$ we can neglect $\exp[w_{X1}t]$. The last equation becomes:

$$\begin{aligned}
\partial_t E &= w_{E2} E_2 (e^{w_{E2}t} - e^{w_{E1}t}) \approx w_{E2} E_2 e^{w_{E2}t} \\
&= A \partial_t p = \left(1 + A_1 (e^{w_{A1}t} - 1) \right) \\
&\quad + A_2 (e^{w_{A2}t} - 1) w_{p2} p_2 (e^{w_{p2}t} - e^{w_{p1}t}) \\
&\approx \left(1 + \frac{\beta}{w_{A1}\tau_B} f + A_2 (w_{A2}t) \right) w_{p2} p_2 e^{w_{p2}t}
\end{aligned} \tag{B3}$$

For arbitrary t this relation can only hold if $w_{E2} = w_{p2}$. Thus we define:

$$w \equiv w_{E2} = w_{p2}$$

Let us also get an estimate for w_{A1} , the smaller exponent in A . We will go very close to the transition line where $w_{X2} \approx 0$. From Eq. (1) we have:

$$\frac{(\tau_B \partial_t + 1) \partial_t A}{A} = \beta \frac{\partial_t E}{E} \approx \beta \frac{w_{E2} e^{wt}}{E} \approx 0$$

With an exponential ansatz the left hand side is:

$$(\tau_B w_A + 1) w_A = 0$$

The greater root is $w_{A2} = 0$ and the smaller root is $w_{A1} = 1/\tau_B$. Even away from the transition line we approximately have:

$$w_{A1} + w_{A2} \approx \frac{1}{\tau_B}$$

Thus we can approximate the expression for A_1 to:

$$A_1 \approx \frac{\beta}{\tau_B} \ln(1+f) \approx \beta f$$

This way Eq. (3) becomes

$$E_2 \approx (1 + \beta f + A_2 (w_{A2}t)) p_2 \approx (1 + \beta f) p_2 \tag{B4}$$

Eq. (1) becomes

$$\frac{(\tau_B w_{A2} + 1) w_{A2} A_2 e^{w_{A2}t}}{1 + \beta f (e^{t/\tau_B} - 1) + A_2 w_{A2}t} \approx \beta \frac{w_{E2} e^{wt}}{1 + f + E_2 w t} \tag{B5}$$

Which again only holds if $w_{A2} = w$. Thus

$$w_{A2} = w_{E2} = w_{p2} = w$$

Again, note that the condition for being close to the transition point was:

$$w \ll \frac{1}{\tau_A} + \frac{1}{\tau_B}$$

Discarding higher than linear order terms in w and looking at times $t/\tau_B \gg 1$ yields:

$$\frac{A_2}{1 - \beta f} = \beta \frac{E_2}{1 + f} + O(w) \tag{B6}$$

Performing the same procedure on Eq. (2) results in (since $w \ll \frac{1}{\tau_{A,B}}$)

$$\begin{aligned}
\frac{(\tau_A w + 1) w p_2 e^{wt}}{1 + p_2 w t} &\approx \alpha \frac{w A_2 e^{wt}}{1 + \beta f (e^{t/\tau_B} - 1) + A_2 w_{A2}t} \\
p_2 &= \alpha \frac{A_2}{1 - \beta f} + O(w) = \alpha \beta \frac{E_2}{(1 + f)} + O(w) \\
&\approx \gamma \frac{1 + \beta f}{1 + f} p_2 + O(w)
\end{aligned} \tag{B7}$$

And so, the condition for the phase transition becomes:

$$\gamma = \frac{1 + f}{1 - \beta f}$$

Now, taking the shock to zero $f \rightarrow 0$ results in a phase transition at:

$$\text{Phase Transition at: } \gamma = 1 \tag{B8}$$